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Limiting distribution of the continuity modulus for Gaussian processes with stationary increments

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Abstract

Let $\{X(t), t \in \mathbb{R}\}$ be a Gaussian process with stationary increments, zero mean and a.s. continuous paths, whose variogram $\gamma(t)$ behaves like $c|t|^\alpha$, $c > 0$, $\alpha \in (0, 2)$, as $t \rightarrow 0$. We show that the continuity modulus of X has asymptotically Gumbel distribution. In the case $\alpha = 2$, a non-Gumbel limiting distribution is obtained.

Key words: Lévy's continuity modulus; Gaussian processes; Extremes.

1 Introduction

Let $\{X(t), t \in \mathbb{R}\}$ be a stochastic process. The continuity modulus $\{\omega_n, n > 0\}$ of X on the interval $[0, 1]$ is defined as

$$\omega_n = \sup_{t \in [0, 1]} \left(X\left(t + \frac{1}{n}\right) - X(t) \right).$$

If X is a standard Brownian motion, then a classical theorem of Lévy says that

$$\lim_{n \rightarrow \infty} \omega_n / \sqrt{2n^{-1} \log n} = 1 \text{ a.s.}$$

A more precise result on the lim sup behavior of ω_n is the integral test of Chung et al. (1959); for the lim inf behavior see Révész (1982). Lévy's continuity modulus was generalized from Brownian motion to Gaussian processes with stationary increments in Marcus (1968), Marcus (1970), Sirao and Watanabe (1968), Sirao and Watanabe (1970).

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Here, we are interested in the properties of the *distribution* of ω_n as $n \rightarrow \infty$. To formulate our result we assume that

- (A) $\{X(t), t \in \mathbb{R}\}$ is a Gaussian process with stationary increments, zero mean and a.s. continuous paths.

Recall that X is said to have stationary increments if the law of the process $\{X(t + t_0) - X(t_0), t \in \mathbb{R}\}$ does not depend on the choice of $t_0 \in \mathbb{R}$. We denote by $\gamma(t) = \mathbb{E}[(X(t) - X(0))^2]$ the so-called variogram of X . Note that a stationary Gaussian process with zero mean and covariance $r(t)$ has stationary increments, the variogram being $\gamma(t) = 2(r(0) - r(t))$.

We suppose that there exist $\alpha \in (0, 2)$ and $c > 0$ such that

- (A1) $\gamma(t) = c|t|^\alpha + o(|t|^\alpha)$ as $t \rightarrow 0$.
 (A2) γ is two times differentiable on $(0, 1)$ and $\gamma''(t) < Kt^{\alpha-2}$ for some $K > 0$ and all $t \in (0, 1)$.

Note that the above conditions are satisfied for the fractional Brownian motion having $\gamma(t) = |t|^\alpha$, as well as for the generalized Ornstein-Uhlenbeck process and the generalized Cauchy model, the latter two being stationary Gaussian processes having the covariance functions $r_{OU}(t) = e^{-|t|^\alpha}$, $r_{Cauchy}(t) = (1 + |t|^\alpha)^{-\beta}$ and variograms $\gamma_{OU}(t) = 2(1 - e^{-|t|^\alpha})$, $\gamma_{Cauchy}(t) = 2(1 - (1 + |t|^\alpha)^{-\beta})$, where $\alpha \in (0, 2)$, $\beta > 0$. Another example is given by $\gamma(t) = \log(1 + |t|^\alpha)$, $\alpha \in (0, 2)$.

For $\tau \in \mathbb{R}$ and a constant $G_\alpha > 0$ whose exact value will be not important for us, define

$$u_n(\tau) = (2 \log n)^{\frac{1}{2}} + (2 \log n)^{-\frac{1}{2}} \left(\frac{2 - \alpha}{\alpha} \log \log n + \log G_\alpha + \tau \right). \quad (1)$$

Theorem 1 *Suppose that conditions (A), (A1), (A2) are satisfied. Then for each $\tau \in \mathbb{R}$ we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}[\omega_n \leq \sqrt{\gamma(1/n)} u_n(\tau)] = \exp(-e^{-\tau}).$$

The above theorem is valid for $\alpha \in (0, 2)$. For $\alpha = 2$ the situation is quite different.

Theorem 2 *Suppose that X satisfies (A). Suppose further that γ has a Lipschitz continuous second derivative and set $c = \gamma''(0)$. Then there is a stationary Gaussian process ξ with zero mean and covariance $\mathbb{E}[\xi(0)\xi(t)] = \gamma''(t)/c$. Further, for every $\tau \in \mathbb{R}$ we have*

$$\lim_{n \rightarrow \infty} \mathbb{P}[\omega_n \leq \sqrt{\gamma(1/n)} \tau] = \mathbb{P}[\sup_{t \in [0,1]} \xi(t) \leq \tau].$$

2 Extremes of Gaussian processes

Our proofs are based on results about extremes of Gaussian processes. First, we recall a fundamental theorem due to Pickands (1969a), Pickands (1969b), see also Theorem 12.3.5 of Leadbetter et al. (1983).

Theorem 3 *Let $\{X(t), t \in \mathbb{R}\}$ be a stationary Gaussian process with zero mean, covariance function $r(t) = \mathbb{E}[X(0)X(t)]$ and a.s. continuous paths. Suppose that for some $\alpha \in (0, 2]$ the following conditions hold*

- (B1) $r(t) = 1 - |t|^\alpha + o(|t|^\alpha)$ as $t \rightarrow 0$.
- (B2) $r(t) = 1$ iff $t = 0$.
- (B3) $r(t) = o(1/\log |t|)$ as $t \rightarrow \infty$.

Then, with $u_n(\tau)$ defined by (1),

$$\lim_{n \rightarrow \infty} \mathbb{P}[\sup_{t \in [0, n]} X(t) \leq u_n(\tau)] = \exp(-e^{-\tau}).$$

Pickands' theorem was generalized in various directions. Here, we need the following result about extremes of *sequences* of stationary Gaussian processes due to Seleznev (1991).

Theorem 4 *For $n > 0$ let $\{\xi_n(t), t \in \mathbb{R}\}$ be a stationary Gaussian process with zero mean, covariance function $r_n(t)$, and a.s. continuous paths. Suppose that*

- (C1) $r_n(t) = 1 - c_n |t|^\alpha + \varepsilon_n(t) |t|^\alpha$, where c_n and $\varepsilon_n(t)$ satisfy $\lim_{n \rightarrow \infty} c_n = 1$ and, uniformly in n , $\lim_{t \rightarrow 0} \varepsilon_n(t) = 0$.
- (C2) for every $\varepsilon > 0$ we have $\sup\{r_n(t) : n > 0, t \in [\varepsilon, n]\} < 1$.
- (C3) for every $\varepsilon > 0$ there is $T(\varepsilon) > 0$ such that $r_n(t) \log t < \varepsilon$ for every $n > 0$, $t \in [T(\varepsilon), n]$.

Let $u_n(\tau)$ be defined by (1). Then

$$\lim_{n \rightarrow \infty} \mathbb{P}[\sup_{t \in [0, n]} \xi_n(t) \leq u_n(\tau)] = \exp(-e^{-\tau}).$$

3 Proofs of Theorem 1 and Theorem 2

Before proving Theorem 1 in its full generality, we consider an instructive special case. Suppose that X is a fractional Brownian motion with index $\alpha \in (0, 2)$, in which case $\gamma(t) = |t|^\alpha$. For $n > 0$ define a stationary Gaussian process

$\{\xi_n(t), t \in \mathbb{R}\}$ by

$$\xi_n(t) = \frac{X(\frac{t+1}{n}) - X(\frac{t}{n})}{\sqrt{\gamma(1/n)}}. \quad (2)$$

Note that the law of ξ_n does not depend on n due to the self-similarity property of the fractional Brownian motion. The covariance function of $\xi_n(t)$ is given by $r(t) = \frac{1}{2}(|1-t|^\alpha + |1+t|^\alpha) - |t|^\alpha$ and can be easily seen to satisfy conditions (B1), (B2), (B3). Thus, by Pickands' Theorem 3,

$$\lim_{n \rightarrow \infty} \mathbb{P}[\omega_n \leq \sqrt{\gamma(1/n)}u_n(\tau)] = \lim_{n \rightarrow \infty} \mathbb{P}[\sup_{t \in [0, n]} \xi_n(t) \leq u_n(\tau)] = \exp(-e^{-\tau}),$$

which proves Theorem 1 if X is a fractional Brownian motion.

Proof of Theorem 1. We suppose that X is a Gaussian process satisfying (A), (A1), (A2). Define ξ_n as above by (2) and note that this time the law of ξ_n may depend on n . We show that the sequence ξ_n satisfies the conditions of Theorem 4. Note that ξ_n is stationary with $\mathbb{E}[\xi_n(t)] = 0$, $\mathbb{E}[\xi_n(t)^2] = 1$. Using the identity

$$\mathbb{E}[X(t_1)X(t_2)] = \frac{1}{2}(\mathbb{E}[X(t_1)^2] + \mathbb{E}[X(t_2)^2] - \gamma(t_1 - t_2))$$

together with (2), it is easy to see that the covariance function $r_n(t) = \mathbb{E}[\xi_n(0)\xi_n(t)]$ is given by

$$r_n(t) = \frac{1}{2\gamma(1/n)} \left[\gamma\left(\frac{1-t}{n}\right) + \gamma\left(\frac{1+t}{n}\right) - 2\gamma\left(\frac{t}{n}\right) \right]. \quad (3)$$

Using (A1), we may write $\gamma(\frac{t}{n}) = c|\frac{t}{n}|^\alpha + |\frac{t}{n}|^\alpha \delta(\frac{t}{n})$, where $\lim_{s \rightarrow 0} \delta(s) = 0$. Then

$$r_n(t) = 1 - c_n|t|^\alpha + \varepsilon_n(t)|t|^\alpha,$$

where $c_n = c/(n^\alpha \gamma(1/n))$ and

$$\varepsilon_n(t) = \frac{1}{2\gamma(1/n)|t|^\alpha} \left[\gamma\left(\frac{1-t}{n}\right) + \gamma\left(\frac{1+t}{n}\right) - 2\gamma\left(\frac{1}{n}\right) \right] - \frac{\delta(t/n)}{n^\alpha \gamma(1/n)}.$$

Note that by (A1) we have $\lim_{n \rightarrow \infty} c_n = 1$. To show that $\lim_{t \rightarrow 0} \varepsilon_n(t) = 0$ uniformly in n , we may assume that $t \in (0, 1/2)$. By the mean value theorem for the second order difference quotient, there is $s = s(n, t)$ in the interval $[\frac{1-t}{n}, \frac{1+t}{n}]$ such that

$$\varepsilon_n(t) = \frac{1}{2\gamma(1/n)|t|^\alpha} \frac{t^2}{n^2} \gamma''(s) - \frac{\delta(t/n)}{n^\alpha \gamma(1/n)} = \frac{\gamma''(s)n^{\alpha-2}}{2n^\alpha \gamma(1/n)} |t|^{2-\alpha} - \frac{\delta(t/n)}{n^\alpha \gamma(1/n)}.$$

Now, by (A1), $n^\alpha \gamma(1/n)$ remains bounded away from zero as $n \rightarrow \infty$. Furthermore, we have $s \geq 1/(2n)$ and thus, by (A2),

$$\gamma''(s)n^{\alpha-2} \leq \gamma''(s)(1/(2s))^{\alpha-2} \leq 2^{2-\alpha} K.$$

It follows that $\lim_{t \rightarrow 0} \varepsilon_n(t) = 0$ uniformly in n , which shows that condition (C1) of Theorem 4 is satisfied. We show that (C3) holds. Fix $\varepsilon > 0$. We may assume that $t > 1$. By the mean value theorem for the second order difference quotient, we have, for some $s = s(n, t)$ in $[\frac{t-1}{n}, \frac{t+1}{n}]$,

$$r_n(t) = \frac{1}{2\gamma(1/n)} \left[\gamma\left(\frac{t+1}{n}\right) + \gamma\left(\frac{t-1}{n}\right) - 2\gamma\left(\frac{t}{n}\right) \right] = \frac{\gamma''(s)}{2n^2\gamma(1/n)}.$$

First suppose that $t \in [\frac{n}{2}, n]$. Then $s > 1/3$ for large n , and, by (A2), $\gamma''(s)$ is bounded from above. Further, since by (A1) $\gamma(1/n)n^\alpha$ is bounded away from 0, we obtain that for some constants C_1, C_2

$$r_n(t) \leq \frac{C_1}{(\gamma(1/n)n^\alpha)n^{2-\alpha}} \leq \frac{C_2}{n^{2-\alpha}} < \frac{\varepsilon}{\log t},$$

where the last inequality holds if n is sufficiently large. Now take some large T_0 and suppose that $t \in [T_0, \frac{n}{2}]$. By (A2), we have $\gamma''(s) \leq C_3(t/n)^{\alpha-2}$. Thus,

$$r_n(t) \leq \frac{C_3 t^{\alpha-2}}{2n^2\gamma(1/n)n^{\alpha-2}} = \frac{C_3}{2\gamma(1/n)n^\alpha} t^{\alpha-2} \leq C_4 t^{\alpha-2} < \frac{\varepsilon}{\log t},$$

where the last inequality holds if T_0 is sufficiently large. This proves (C3). We show that (C2) holds. Fix $\varepsilon > 0$ and let $\delta > 0$. We may suppose that $t \in [\varepsilon, T_0]$, since otherwise (C2) follows from the above proof of (C3). If $n > N_0$ is big enough, then it follows from (A1) that with $c_* = c - \delta$ and $c^* = c + \delta$

$$c_*(t/n)^\alpha < \gamma(t/n) < c^*(t/n)^\alpha.$$

Analogous inequalities hold also for $\gamma(\frac{1}{n})$, $\gamma(\frac{t+1}{n})$ and $\gamma(\frac{t-1}{n})$. Thus, by (3),

$$\begin{aligned} r_n(t) &< \frac{1}{2c_*n^{-\alpha}} \left(c^* \frac{|t+1|^\alpha}{n^\alpha} + c^* \frac{|t-1|^\alpha}{n^\alpha} - 2c_* \frac{t^\alpha}{n^\alpha} \right) \\ &= \frac{c^*}{2c_*} (|1+t|^\alpha + |1-t|^\alpha) - t^\alpha. \end{aligned}$$

Note that $\sup_{t \in [\varepsilon, T_0]} \frac{1}{2} (|1+t|^\alpha + |1-t|^\alpha) - t^\alpha < 1$. Thus, for sufficiently small δ , we have $\sup_{t \in [\varepsilon, T_0]} \frac{c^*}{2c_*} (|1+t|^\alpha + |1-t|^\alpha) - t^\alpha < 1$ and hence $\sup_{t \in [\varepsilon, T_0], n > N_0} r_n(t) < 1$. This proves (C2).

Applying Theorem 4 to the sequence ξ_n , we obtain

$$\lim_{n \rightarrow \infty} \mathbb{P}[\omega_n \leq \sqrt{\gamma(1/n)} u_n(\tau)] = \lim_{n \rightarrow \infty} \mathbb{P}[\sup_{t \in [0, n]} \xi_n(t) \leq u_n(\tau)] = \exp(-e^{-\tau}).$$

This finishes the proof of Theorem 1. \square

Proof of Theorem 2. Let the assumptions of Theorem 2 be satisfied. Then, in

particular, $\gamma(t) = ct^2/2 + o(t^2)$ as $t \rightarrow 0$. For $n > 0$ define a process ξ_n by

$$\xi_n(t) = \frac{X(t + \frac{1}{n}) - X(t)}{\sqrt{\gamma(1/n)}}. \quad (4)$$

Analogously to (3), the covariance function of ξ_n is given by

$$r_n(t) = \frac{1}{2\gamma(1/n)} \left[\gamma(t + \frac{1}{n}) + \gamma(t - \frac{1}{n}) - 2\gamma(t) \right].$$

Thus, for some $s = s(n, t) \in [t - \frac{1}{n}, t + \frac{1}{n}]$,

$$r_n(t) = \frac{\gamma''(s(n, t))}{2n^2\gamma(1/n)} \rightarrow \frac{\gamma''(t)}{c}, \quad n \rightarrow \infty.$$

It follows that $\gamma''(\cdot)/c$ is a covariance function of some stationary Gaussian process ξ . Since γ'' is supposed to be Lipschitz continuous, we may choose a version of the process ξ with continuous sample paths. We also take ξ to have zero mean. Then it follows that the finite-dimensional distributions of ξ_n converge to those of ξ .

Now we show that the sequence of processes $\{\xi_n(t), t \in [0, 1]\}_{n>0}$ converges to the process $\{\xi(t), t \in [0, 1]\}$ in the sense of weak convergence on $C[0, 1]$, the space of continuous functions on $[0, 1]$. We need only to show that the sequence ξ_n is tight. Using the usual mean value theorem twice, we obtain for some $s_1 = s_1(n, t) \in [t - \frac{1}{n}, t]$ and $s_2 = s_2(n, t) \in [t, t + \frac{1}{n}]$

$$r'_n(t) = \frac{1}{2\gamma(1/n)} \left[\gamma'(t + \frac{1}{n}) + \gamma'(t - \frac{1}{n}) - 2\gamma'(t) \right] = \frac{\gamma''(s_2) - \gamma''(s_1)}{2n\gamma(1/n)}.$$

Using the Lipschitz continuity of γ'' , we obtain for some $L, C > 0$

$$r'_n(t) \leq L|s_2 - s_1|/(2n\gamma(1/n)) \leq L/(n^2\gamma(1/n)) \leq C.$$

It follows that for any $t, s \in [0, 1]$

$$\mathbb{E}[(\xi_n(t) - \xi_n(s))^2] = 2 - 2r_n(t - s) = 2(r_n(0) - r_n(t - s)) \leq 2C|t - s|.$$

Together with the fact that $\xi_n(0)$ is standard Gaussian for every n , this implies the tightness of the family ξ_n , see e.g. Corollary 11.7 of Ledoux and Talagrand (1991). It follows that ξ_n converges to ξ weakly on $C[0, 1]$ and hence

$$\lim_{n \rightarrow \infty} \mathbb{P}[\omega_n \leq \sqrt{\gamma(1/n)}\tau] = \lim_{n \rightarrow \infty} \mathbb{P}[\sup_{t \in [0, 1]} \xi_n(t) \leq \tau] = \mathbb{P}[\sup_{t \in [0, 1]} \xi(t) \leq \tau].$$

This finishes the proof of Theorem 2. □

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